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Initial value problem for the dynamic system of anisotropic elasticity

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Abstract

The main object of this paper is the Cauchy problem for the dynamic system of anisotropic elasticity. Existence and uniqueness theorems of weak and smooth solutions of this problem are established by the reduction of the original elasticity system into a symmetric hyperbolic system of the first order. The numerical method of the Cauchy problem solving for anisotropic elastic system with polynomial data is obtained and its correctness is established. The simulations of the numerical solutions are presented.

Keywords: Anisotropic elasticity; Dynamic system; Cauchy problem; Simulation

1. Introduction

A dynamic mathematical model of wave propagations in anisotropic media is described by a system of partial differential equations (Dieulesaint and Royer, 1980; Fedorov, 1968; Ting, 1996; Ting et al., 1990). As is well known the well-posedness of an initial value problem (IVP) is a basic consistency check of the system for which IVP is considered. The goal to realize that a system of partial differential equations is reducible to a symmetric hyperbolic system of the first order is natural because in this case there is a chance to show that this system has a well-posed IVP. The different approaches to analyze systems of elasticity and magnetoelasticity as hyperbolic systems may be found in the literature (Beig and Schmidt, 2003; Christodoulon, 2000; Cohen and Fauqueux, 2003; Duff, 1960; Marsden and Hughes, 1983; Sacks and Yakhno, 1998; Yakhno, 1998 and others).

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Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We assume that $\mathbb{R}^3$ is an elastic medium, whose small amplitude vibrations

$$\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \tag{1}$$

are governed by the system of partial differential equations and initial conditions

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \ t > 0, \ j = 1, 2, 3, \tag{2}$$

$$u_j(x,0) = \varphi_j(x), \quad \left. \frac{\partial u_j(x,t)}{\partial t} \right|_{t=0} = \psi_j(x), \ j = 1, 2, 3. \tag{3}$$

Here ρ is the density of the medium,

$$\sigma_{jk} = \sum_{l,m=1}^{3} C_{jklm} \epsilon_{lm}, \quad j,k = 1, 2, 3 \tag{4}$$

is the stress tensor,

$$\epsilon_{lm} = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l}\right), \quad l,m = 1, 2, 3 \tag{5}$$

is the strain tensor, and $\{C_{jklm}\}_{j,k,l,m=1}^{3}$ are the elastic moduli of the medium, $\varphi_j(x)$, $\psi_j(x)$, $j = 1, 2, 3$ are smooth functions. We assume that ρ and C_{jklm} are constants.

It is convenient and customary to describe the elastic moduli in terms of a 6×6 matrix according to the following conventions relating a pair (j,k) of indices $j,k = 1, 2, 3$ to a single index $\alpha = 1, \ldots, 6$:

$$(1,1) \leftrightarrow 1, \quad (2,2) \leftrightarrow 2, \quad (3,3) \leftrightarrow 3, \quad (2,3),(3,2) \leftrightarrow 4, \quad (1,3),(3,1) \leftrightarrow 5, \quad (1,2),(2,1) \leftrightarrow 6. \tag{6}$$

This correspondence is possible due to the symmetry properties $C_{jklm} = C_{kjlm} = C_{jkml}$. The additional symmetry property $C_{jklm} = C_{lmjk}$ implies that the matrix

$$\mathbf{C} = (C_{\alpha\beta})_{6\times 6} \tag{7}$$

of all moduli where $\alpha = (jk)$, $\beta = (lm)$, is symmetric. We will assume also that $\rho > 0$ and the matrix $(C_{\alpha\beta})_{6\times 6}$ is positive definite.

In this paper we analyze relations (2)–(5) from two points of view. The first one is the following. The equalities (2) and (4) are written in the form of a symmetric hyperbolic system of the first order. Using the theory of the Cauchy problems for symmetric hyperbolic systems (Mizohata, 1973) we obtain the general theoretical results: existence and uniqueness theorems of weak and smooth solutions of the initial value problem (2) and (3). Sections 2 and 3 contain these results. The second view point is related with the analysis of relations (2) and (4) as the Cauchy problem for the second order hyperbolic equations system with the polynomial initial data. As a result of this analysis we show in Section 4 that the solution of (2) and (3) is a polynomial with respect to the lateral variables if the initial data are polynomials relative to the same variables. To find a solution of the initial value problem (2) and (3) with polynomial data we specified a procedure of the polynomial coefficients recovery. This procedure we called polynomial solution method (PS-method). This method is described in Section 5. The correctness of this method and simulations of the numerical solutions of the Cauchy problems for anisotropic elastic system are given in Section 6. In conclusion some generalizations and remarks on our research are described. At the end of the paper there are appendices containing facts from the theory of symmetric hyperbolic systems of the first order and matrix theory in the form and volume that are convenient for us.

2. System of elasticity as a symmetric hyperbolic system

We show here that relations (2) and (4) may be written as a symmetric hyperbolic system.

Let

$$U_i = \frac{\partial u_i}{\partial t}, \quad i = 1,2,3. \tag{8}$$

Using the rule (6) we denote a pair (j,k) of indices $j,k = 1,2,3$ as a single index α, $\alpha = 1,2,\ldots,6$. This correspondence is possible to make for σ_{jk} and ϵ_{jk} because of the symmetry properties $\sigma_{jk} = \sigma_{kj}$, $\epsilon_{jk} = \epsilon_{kj}$.

We have

$$\mathbf{T} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)^*, \quad \boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6)^*. \tag{9}$$

Here $*$ is the sign of transposition.

Let

$$\mathbf{U} = (U_1, U_2, U_3)^*, \quad \mathbf{Y} = (\epsilon_1, \epsilon_2, \epsilon_3, 2\epsilon_4, 2\epsilon_5, 2\epsilon_6)^*. \tag{10}$$

Consider now system (2). Using vectors $\mathbf{U}$ and $\mathbf{T}$ this system may be written in another form as follows:

$$\rho \frac{\partial \mathbf{U}}{\partial t} + \sum_{k=1}^{3} \mathbf{A}_k^1 \frac{\partial \mathbf{T}}{\partial x_k} = 0, \tag{11}$$

where

$$\mathbf{A}_1^1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{A}_2^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_3^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Consider the relation (4). Using vectors $\mathbf{T}$ and $\mathbf{Y}$ this relation can be written as

$$\mathbf{T} = \mathbf{C}\mathbf{Y}, \tag{13}$$

where $\mathbf{C}$ is defined by (7). Differentiating (13) with respect to t and multiplying both sides by $\mathbf{C}^{-1}$ we find

$$\mathbf{C}^{-1} \frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \mathbf{Y}}{\partial t}, \tag{14}$$

where $\mathbf{C}^{-1}$ is the inverse matrix to $\mathbf{C}$. Using the relations

$$\begin{aligned} &\frac{\partial \epsilon_j}{\partial t} = \frac{\partial U_j}{\partial x_j}, \quad j = 1,2,3; \qquad 2\frac{\partial \epsilon_4}{\partial t} = \frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2}, \\ &2\frac{\partial \epsilon_5}{\partial t} = \frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1}, \qquad 2\frac{\partial \epsilon_6}{\partial t} = \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}, \end{aligned} \tag{15}$$

the system (14) may be written as

$$\mathbf{C}^{-1} \frac{\partial \mathbf{T}}{\partial t} + \sum_{j=1}^{3} (\mathbf{A}_j^1)^* \frac{\partial \mathbf{U}}{\partial x_j} = 0, \tag{16}$$

where $(\mathbf{A}_j^1)^*$, $j = 1,2,3$ are adjoint matrices to $\mathbf{A}_j^1$, $j = 1,2,3$.

The relations (11) and (16) can be presented by a single system as

$$\mathbf{A}_0 \frac{\partial \mathbf{V}}{\partial t} + \sum_{j=1}^{3} \mathbf{A}_j \frac{\partial \mathbf{V}}{\partial x_j} = 0, \tag{17}$$

where $\mathbf{A}_0 = \mathrm{diag}(\rho\mathbf{I}_3, \mathbf{C}^{-1})$ is the block-diagonal matrix,

$$\mathbf{V} = \begin{pmatrix} \mathbf{U} \\ \mathbf{T} \end{pmatrix}, \quad \mathbf{A}_j = \begin{bmatrix} \mathbf{0}_{3,3} & \mathbf{A}_j^1 \\ \mathbf{A}_j^{1*} & \mathbf{0}_{6,6} \end{bmatrix}, \quad j = 1,2,3. \tag{18}$$

Here and further $\mathbf{I}_m$ is the unit matrix of the order $m \times m$ and $\mathbf{0}_{l,m}$ is the zero matrix of the order $l \times m$.

To show that the system (17) is transformed into the following form:

$$\mathbf{I}_9 \frac{\partial \tilde{\mathbf{V}}}{\partial t} + \sum_{j=1}^{3} \tilde{\mathbf{A}}_j \frac{\partial \tilde{\mathbf{V}}}{\partial x_j} = 0, \tag{19}$$

where $\tilde{\mathbf{A}}_j$, $j = 1,2,3$ are symmetric, we use two facts of matrix theory. The first one says that for the symmetric positive definite matrix $\mathbf{C}$ there exists a symmetric positive definite matrix $\mathbf{M}$ such that $\mathbf{C}^{-1} = \mathbf{M}^2$ (see Theorems 6, 7 of Appendix A). The second fact explains that the matrix $\mathbf{M}^{-1}$, which is inverse to $\mathbf{M}$, is symmetric (see Theorem 6 of Appendix A). Considering now (17) and multiplying it by the matrix

$$\mathbf{S} = \begin{bmatrix} \rho^{-\frac{1}{2}}\mathbf{I}_3 & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3} & \mathbf{M}^{-1} \end{bmatrix} \tag{20}$$

from the left-hand side we find (19), where

$$\tilde{\mathbf{A}}_j = \mathbf{S}\mathbf{A}_j\mathbf{S}, \quad \mathbf{V} = \mathbf{S}\tilde{\mathbf{V}}. \tag{21}$$

We note that $\tilde{\mathbf{A}}_j$, $j = 1,2,3$ are still symmetric (see Theorem 8 of Appendix A).

3. Existence and uniqueness theorems for (2) and (3)

In this section the existence and uniqueness theorems for the weak and smooth solutions of the initial value problem (2) and (3) are described. These results follow from the reduction of (2) to the symmetric hyperbolic system (19) and the theory of symmetric hyperbolic systems (Evans, 1998; Mizohata, 1973), (see also Appendix B).

Using notations and reasoning of the Sections 1, 2 and equalities

$$U_i(x,0) = \psi_i(x), \quad \sigma_{jk}|_{t=0} = \sum_{l,m=1}^{3} C_{jklm} \frac{\partial \varphi_l(x)}{\partial x_m}, \quad i,j,k = 1,2,3, \tag{22}$$

we obtain from (2)–(5) the initial value problem for (19) with known data

$$\tilde{\mathbf{V}}(x,0) = \mathbf{V}^0(x), \quad x \in \mathbb{R}^3. \tag{23}$$

The theory of symmetric hyperbolic systems of the first order gives a chance to get the existence and uniqueness theorems for the Cauchy problem (19) and (23). We describe these existence and uniqueness theorems for weak and smooth solutions in Appendix B in a form which is convenient for us (see Theorems 9–11). These theorems may be written in terms of the original problem (2) and (3) as we state below. For statements of these theorems we will use the following notation. The spaces $\mathscr{L}^2(\mathbb{R}^3;\mathbb{R}^3)$, $\mathscr{C}^k(\mathbb{R}^3;\mathbb{R}^3)$, $\mathscr{H}^k(\mathbb{R}^3;\mathbb{R}^3)$, $(k = 0,1,2,\ldots)$ consist of all vector functions $\mathbf{w} = (w_1, w_2, w_3)$ such that w_j belongs to $\mathscr{L}^2(\mathbb{R}^3)$, $\mathscr{C}^k(\mathbb{R}^3)$, $\mathscr{H}^k(\mathbb{R}^3)$, $j = 1,2,3$, respectively. Here $\mathscr{C}^k(\mathbb{R}^3)$ is the space of all k times continuously differentiable functions; $\mathscr{H}^k(\mathbb{R}^3)$ is Sobolev space (Evans, 1998), (see also Appendix B) and $\mathscr{L}^2(\mathbb{R}^3) = \mathscr{H}^0(\mathbb{R}^3)$, $\mathscr{C}(\mathbb{R}^3) = \mathscr{C}^0(\mathbb{R}^3)$. The spaces $\mathscr{C}^l([0,T];\mathscr{H}^k(\mathbb{R}^3;\mathbb{R}^3))$, $\mathscr{C}^l([0,T];\mathscr{C}^k(\mathbb{R}^3;\mathbb{R}^3))$ are the spaces of all l times continuously differentiable functions with respect to t such that $\mathbf{u} : [0,T] \to \mathscr{H}^k(\mathbb{R}^3;\mathbb{R}^3)$ and $\mathbf{u} : [0,T] \to \mathscr{C}^k(\mathbb{R}^3;\mathbb{R}^3)$, respectively. Let γ_i, $i = 1,2$ be nonnegative integer numbers,

$$\gamma = (\gamma_1, \gamma_2, 0), \quad |\gamma| = \gamma_1 + \gamma_2, \quad D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2}}. \tag{24}$$

The following existence and uniqueness theorem for weak solution of (2) and (3) is obtained from symmetric hyperbolic system theory of the first order (see Theorem 9 of Appendix B) which we apply to the Cauchy problem (19) and (23).

Theorem 1. *Let* $\boldsymbol{\varphi}(x) \in \mathscr{H}^2(\mathbb{R}^3;\mathbb{R}^3), \boldsymbol{\psi}(x) \in \mathscr{H}^1(\mathbb{R}^3;\mathbb{R}^3)$, *T be a fixed positive number*; $\rho, \{C_{jklm}\}_{j,k,l,m=1}^3$ *satisfy conditions mentioned in Section* 1. *Then there exists a unique weak solution* $\mathbf{u}(x,t)$ *of the initial value problem* (2) *and* (3) *such that*

$$\begin{aligned} &\mathbf{u}(x,t) \in \mathscr{C}^1([0,T];\mathscr{H}^1(\mathbb{R}^3;\mathbb{R}^3)) \bigcap \mathscr{C}^2([0,T];\mathscr{L}^2(\mathbb{R}^3;\mathbb{R}^3)), \\ &\sigma_{jk} = \sum_{l,m=1}^3 C_{jklm} \frac{\partial u_l(x,t)}{\partial x_m} \in \mathscr{C}([0,T];\mathscr{H}^1(\mathbb{R}^3)) \bigcap \mathscr{C}^1([0,T];\mathscr{L}^2(\mathbb{R}^3)). \end{aligned} \tag{25}$$

The following theorem is found from the Theorem 11 in Appendix B which we apply to (19) and (23).

Theorem 2. *Let* $\gamma = (\gamma_1, \gamma_2, 0)$ *be an arbitrary multiindex, T be a fixed positive number*; $\rho, \{C_{jklm}\}_{j,k,l,m=1}^3$ *satisfy conditions mentioned in Section* 1; $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ *be vector functions such that* $D^\gamma \boldsymbol{\varphi}(x) \in \mathscr{H}^4(\mathbb{R}^3;\mathbb{R}^3), D^\gamma \boldsymbol{\psi}(x) \in \mathscr{H}^3(\mathbb{R}^3;\mathbb{R}^3)$. *Then the weak solution* $\mathbf{u}(x,t)$ *of* (2) *and* (3) *satisfies the following properties*:

$$\begin{aligned} &D^\gamma \mathbf{u}(x,t) \in \mathscr{C}^1([0,T];\mathscr{C}^1(\mathbb{R}^3;\mathbb{R}^3)) \bigcap \mathscr{C}^2([0,T];\mathscr{C}(\mathbb{R}^3;\mathbb{R}^3)), \\ &D^\gamma \sigma_{jk} = \sum_{l,m=1}^3 C_{jklm} \frac{\partial D^\gamma u_l}{\partial x_m} \in \mathscr{C}([0,T];\mathscr{C}^1(\mathbb{R}^3)) \bigcap \mathscr{C}^1([0,T];\mathscr{C}(\mathbb{R}^3)), \end{aligned} \tag{26}$$

where D^γ *is defined by* (24).

Applying Theorem 2 for the cases $\gamma = (\gamma_1, \gamma_2, 0)$, $\gamma_1 + \gamma_2 = 0, 1, 2, 3, \ldots$ we obtain the following theorem.

Theorem 3. *Let* $\boldsymbol{\varphi}(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x))$, $\boldsymbol{\psi}(x) = (\psi_1(x), \psi_2(x), \psi_3(x))$ *satisfy*

$$\begin{aligned} &\varphi_j(x) \in \mathscr{C}^\infty_{x_1,x_2}(\mathbb{R}^2;\mathscr{C}^2(\mathbb{R})) \bigcap \mathscr{H}^4(\mathbb{R}^3), \\ &\psi_j(x) \in \mathscr{C}^\infty_{x_1,x_2}(\mathbb{R}^2;\mathscr{C}^1(\mathbb{R})) \bigcap \mathscr{H}^3(\mathbb{R}^3), j = 1,2,3, \end{aligned} \tag{27}$$

then the weak solution $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ *of* (2) *and* (3) *is a classical solution and*

$$u_k(x,t) \in \mathscr{C}^\infty_{x_1,x_2}(\mathbb{R}^2;\mathbb{H}), \quad k = 1,2,3, \tag{28}$$

$$\sigma_{jk} \in \mathscr{C}^\infty_{x_1,x_2}(\mathbb{R}^2;\mathbb{H}), \quad j,k = 1,2,3, \tag{29}$$

where

$$\mathbb{H} = \mathscr{C}^1([0,T];\mathscr{C}^1(\mathbb{R})) \bigcap \mathscr{C}^2([0,T];\mathscr{C}(\mathbb{R})). \tag{30}$$

The following theorem is related to the uniqueness of the solution for the initial value problem inside the conoid of the dependence. Appendix C contains the definition of the dependence conoid and its properties.

Theorem 4. *Let T be a positive number*, $x_0 \in \mathbb{R}^3$ *be an arbitrary point*, $P = (x_0, T) \in \mathbb{R}^4$, $\Gamma(P)$ *be the conoid of the dependence for the system* (19), $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ *be a solution of the initial value problem* (2) *and* (3) *such that*

$$u_j(x,t) \in \mathscr{C}^1([0,T];\mathscr{H}^1(\mathbb{R}^3)) \bigcap \mathscr{C}^2([0,T];\mathscr{L}^2(\mathbb{R}^3)), \tag{31}$$

$$\sigma_{jk} = \sum_{l,m=1}^{3} C_{jklm} \frac{\partial u_l(x,t)}{\partial x_m} \in \mathscr{C}([0,T]; \mathscr{H}^1(\mathbb{R}^3)) \bigcap \mathscr{C}^1([0,T]; \mathscr{L}^2(\mathbb{R}^3)). \tag{32}$$

Then if $\mathbf{u}(x,t)$ *vanishes on* $S(0)$ *it also vanishes on each surface* $S(h)$, $h \in (0,T)$. *Here* $S(0)$ *and* $S(h)$ *are parts of the planes* $t = 0$ *and* $t = h$, *respectively, inside the conoid* $\Gamma(P)$.

The proof of this theorem follows from the reduction of the system (2) into a symmetric hyperbolic system (19) and Theorem 14 of Appendix C.

4. Solution of (2) and (3) with polynomial data with respect to lateral variables

The goal of this section is to show that the solution $\mathbf{u} = (u_1, u_2, u_3)$ of the problem (2) and (3) can be written in the polynomial form with respect to lateral variables x_1, x_2 if the data $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$ have the polynomial form relative to x_1, x_2.

We use the following notations and assumptions in this section. Let T be an arbitrary positive number, $x_0 = (0,0,0)$, $P = (x_0, T) \in \mathbb{R}^4$, $\Gamma(P)$ be the conoid of the dependence for the system (19), $S(0)$ be the bottom of this conoid for $t = 0$. Suppose that the solution $\mathbf{u}(x,t)$ of (2) and (3) satisfies the Theorem 4 and initial data $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$ are such that $\varphi_j(x) \in \mathscr{C}^4(S(0))$, $\psi_j(x) \in \mathscr{C}^3(S(0))$ and for $x \in S(0)$ the following polynomial presentations hold

$$\varphi_j(x) = \sum_{l=0}^{p} \sum_{m=0}^{p} \varphi_j^{l,m}(x_3) x_1^m x_2^l, \quad j = 1,2,3, \tag{33}$$

$$\psi_j(x) = \sum_{l=0}^{p} \sum_{m=0}^{p} \psi_j^{l,m}(x_3) x_1^m x_2^l, \quad j = 1,2,3, \tag{34}$$

where p is a fixed natural number. The functions $\varphi_j(x)$, $\psi_j(x)$, $j = 1,2,3$ are extended for all $x \in \mathbb{R}^3$ such that their extensions satisfy (27).

According to Theorems 1–3 there exists a unique classical solution $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ of (2) and (3) satisfying (28). We note that components of this solution can be presented in the form

$$u_j(x_1,x_2,x_3,t) = \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} U_j^{s,k}(x_3,t) x_1^s x_2^k, \tag{35}$$

where

$$U_j^{s,k}(x_3,t) = \frac{1}{s!k!} \frac{\partial^{s+k}}{\partial x_1^s \partial x_2^k} u_j(x_1,x_2,x_3,t)|_{x_1=x_2=0}, \quad j = 1,2,3; \ \ s = 0,1,2,\ldots; \ \ k = 0,1,2,\ldots \tag{36}$$

Applying the operator D^γ defined by (24) to Eqs. (2) and (3) and denoting

$$u_j^\gamma = D^\gamma u_j, \quad \varphi_j^\gamma = D^\gamma \varphi_j, \quad \psi_j^\gamma = D^\gamma \psi_j, \quad \epsilon_{lm}^\gamma = \frac{1}{2}\left(\frac{\partial u_l^\gamma}{\partial x_m} + \frac{\partial u_m^\gamma}{\partial x_l}\right), \quad \sigma_{jk}^\gamma = \sum_{l,m=1}^{3} C_{jklm} \epsilon_{lm}^\gamma, \tag{37}$$

we find

$$\rho \frac{\partial^2 u_j^\gamma}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{jk}^\gamma}{\partial x_k}, \quad j = 1,2,3, \quad x \in \mathbb{R}^3, \ \ t > 0, \tag{38}$$

$$u_j^\gamma(x,0) = \varphi_j^\gamma(x), \quad j = 1,2,3, \quad x \in \mathbb{R}^3, \tag{39}$$

$$\left.\frac{\partial u_j^\gamma(x,t)}{\partial t}\right|_{t=0} = \psi_j^\gamma(x), \quad j=1,2,3, \quad x \in \mathbb{R}^3. \tag{40}$$

It follows from (33) and (34) that $\varphi_j^\gamma \equiv 0$, $\psi_j^\gamma \equiv 0$ for $x \in S(0)$, $\gamma_1 > p$ or $\gamma_2 > p$. Theorems 1–4 are valid for the problem (38)–(40) because the initial value problem (38)–(40) is similar to (2) and (3). Using Theorems 1–4 we find that $u_j^\gamma(x,t) \equiv 0$ for $(x,t) \in \Gamma(P)$, $j=1,2,3$ if $\gamma_1 > p$ or $\gamma_2 > p$.

This means that functions $U_j^{s,k}(x_3,t)$, $j=1,2,3$, defined by (36) satisfy the relations $U_j^{s,k}(x_3,t)=0$, $j=1,2,3$ for $(x_3,t) \in \Delta(P)$, where

$$\Delta(P) = \{(x_3,t) \in \mathbb{R}^2 : (0,0,x_3,t) \in \Gamma(P)\}. \tag{41}$$

Therefore the solution $\mathbf{u}=(u_1,u_2,u_3)$ of (2) and (3) has the polynomial form

$$u_j(x_1,x_2,x_3,t) = \sum_{k=0}^{p}\sum_{s=0}^{p} U_j^{s,k}(x_3,t)x_1^s x_2^k, \quad (x,t) \in \Gamma(P), \tag{42}$$

where

$$U_j^{s,k}(x_3,t) \in \mathscr{C}^2(\Delta(P)), \quad j=1,2,3; \quad s,k=0,1,2,\ldots,p. \tag{43}$$

As a result we proved the following theorem.

Theorem 5. *Under above mentioned notations and assumptions the solution* $\mathbf{u}=(u_1,u_2,u_3)$ *of the problem* (2) *and* (3) *may be written in the form* (42) *and* (43).

5. Polynomial solution method (PS-method) for (2) and (3)

According to the Theorem 5 the solution of (2) and (3) has the form (42). The explicit formula for $U_j^{p,p}(x_3,t)$, $j=1,2,3$ and recurrence relations for $U_j^{s,k}(x_3,t)$, $j=1,2,3$; $s,k=0,1,2,\ldots,p, s<p, k<p$ will be found in this section. For finding these recurrence relations the system (2) is written in the form

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mathbf{G}\frac{\partial^2\mathbf{u}}{\partial x_1^2} + \mathbf{H}\frac{\partial^2\mathbf{u}}{\partial x_2^2} + \mathbf{B}\frac{\partial^2\mathbf{u}}{\partial x_3^2} + \mathbf{F}\frac{\partial^2\mathbf{u}}{\partial x_1\partial x_2} + \mathbf{D}\frac{\partial^2\mathbf{u}}{\partial x_1\partial x_3} + \mathbf{E}\frac{\partial^2\mathbf{u}}{\partial x_2\partial x_3}, \tag{44}$$

where

$$\mathbf{G} = \begin{bmatrix} c_{11} & c_{16} & c_{15} \\ c_{16} & c_{66} & c_{56} \\ c_{15} & c_{56} & c_{55} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} c_{16}+c_{16} & c_{12}+c_{66} & c_{14}+c_{56} \\ c_{66}+c_{12} & c_{26}+c_{26} & c_{46}+c_{25} \\ c_{56}+c_{14} & c_{25}+c_{46} & c_{45}+c_{45} \end{bmatrix}, \tag{45}$$

$$\mathbf{H} = \begin{bmatrix} c_{66} & c_{26} & c_{46} \\ c_{26} & c_{22} & c_{24} \\ c_{46} & c_{24} & c_{44} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} c_{15}+c_{15} & c_{14}+c_{56} & c_{13}+c_{55} \\ c_{56}+c_{14} & c_{46}+c_{46} & c_{36}+c_{45} \\ c_{55}+c_{13} & c_{45}+c_{36} & c_{35}+c_{35} \end{bmatrix}, \tag{46}$$

$$\mathbf{B} = \begin{bmatrix} c_{55} & c_{45} & c_{35} \\ c_{45} & c_{44} & c_{34} \\ c_{35} & c_{34} & c_{33} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} c_{56}+c_{56} & c_{46}+c_{25} & c_{36}+c_{45} \\ c_{25}+c_{46} & c_{24}+c_{24} & c_{23}+c_{44} \\ c_{45}+c_{36} & c_{44}+c_{23} & c_{34}+c_{34} \end{bmatrix}. \tag{47}$$

Consider the initial value problem (44) and (3) in which $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$, $\boldsymbol{\psi}=(\psi_1,\psi_2,\psi_3)$ satisfy (33) and (34).

5.1. Finding $\mathbf{U}^{p,p}(x_3,t)=(U_1^{p,p}(x_3,t),U_2^{p,p}(x_3,t),U_3^{p,p}(x_3,t))$

Differentiating (44) and (3) p times with respect to x_1 and then p times with respect to x_2 and using the formula (36) we get the system

$$\rho\frac{\partial^2\mathbf{U}^{p,p}}{\partial t^2}=\mathbf{B}\frac{\partial^2\mathbf{U}^{p,p}}{\partial x_3^2},\quad (x_3,t)\in\Delta(P), \tag{48}$$

with initial conditions

$$\mathbf{U}^{p,p}|_{t=0}=\boldsymbol{\varphi}^{p,p}(x_3),\quad \left.\frac{\partial\mathbf{U}^{p,p}}{\partial t}\right|_{t=0}=\boldsymbol{\psi}^{p,p}(x_3),\quad x_3\in L(P), \tag{49}$$

where

$$L(P)=\{x_3\in\mathbb{R}:(x_3,0)\in\Delta(P)\}, \tag{50}$$

$\mathbf{B}$ is defined by (47) and $\boldsymbol{\varphi}^{p,p}(x_3)$, $\boldsymbol{\psi}^{p,p}(x_3)$ are given coefficients of polynomial expansions (33) and (34) of vector functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$. Since $\mathbf{C}$ defined by (7) is a real symmetric positive definite matrix then $\mathbf{B}$ defined by (47) is a real symmetric positive definite matrix also. Hence $\mathbf{B}$ is congruent to a diagonal matrix of its eigenvalues. That is, there exists an orthogonal matrix $\mathbf{Z}$ such that

$$\mathbf{Z}^{-1}\mathbf{B}\mathbf{Z}=\boldsymbol{\Lambda}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}. \tag{51}$$

Because $\mathbf{B}$ is positive definite, real and symmetric its eigenvalues λ_i, $i=1,2,3$ are real and positive.

Setting

$$\mathbf{U}^{p,p}=\mathbf{Z}\mathbf{Y}^{p,p} \tag{52}$$

in (48) we get

$$\rho\frac{\partial^2\mathbf{Z}\mathbf{Y}^{p,p}}{\partial t^2}=\mathbf{B}\frac{\partial^2\mathbf{Z}\mathbf{Y}^{p,p}}{\partial x_3^2},\quad (x_3,t)\in\Delta(P). \tag{53}$$

We multiply left-hand side of (53) by $\mathbf{Z}^{-1}$ to obtain

$$\rho\frac{\partial^2\mathbf{Y}^{p,p}}{\partial t^2}=\boldsymbol{\Lambda}\frac{\partial^2\mathbf{Y}^{p,p}}{\partial x_3^2},\quad (x_3,t)\in\Delta(P). \tag{54}$$

Denoting $v_j=\left(\frac{\lambda_j}{\rho}\right)^{\frac{1}{2}}$, $j=1,2,3$ and using d'Alembert formula we can solve the Cauchy problem for (54) with the following initial data

$$\mathbf{Y}^{p,p}|_{t=0}=\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p}(x_3),\quad \left.\frac{\partial\mathbf{Y}^{p,p}}{\partial t}\right|_{t=0}=\mathbf{Z}^{-1}\boldsymbol{\psi}^{p,p}(x_3),\ \ x_3\in L(P). \tag{55}$$

The explicit solution of (54) and (55) is given by

$$Y_j^{p,p}(x_3,t)=\frac{1}{2}\Big[(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p})_j(x_3-v_jt)+(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p})_j(x_3+v_jt)\Big]+\frac{1}{2v_j}\int_{x_3-v_jt}^{x_3+v_jt}(\mathbf{Z}^{-1}\boldsymbol{\psi}^{p,p})_j(\sigma)\mathrm{d}\sigma,$$
$$j=1,2,3,\ \ (x_3,t)\in\Delta(P). \tag{56}$$

Using (52) and (56) we find

$$U_i^{p,p}(x_3,t)=\sum_{j=1}^{3}Z_{ij}\left\{\frac{1}{2}\Big[(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p})_j(x_3-v_jt)+(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p})_j(x_3+v_jt)\Big]+\frac{1}{2v_j}\int_{x_3-v_jt}^{x_3+v_jt}(\mathbf{Z}^{-1}\boldsymbol{\psi}^{p,p})_j(\sigma)\mathrm{d}\sigma\right\},$$
$$i=1,2,3,\ \ (x_3,t)\in\Delta(P), \tag{57}$$

where Z_{ij}, $i,j = 1,2,3$ are elements of the matrix $\mathbf{Z}$.

5.2. Finding $\mathbf{U}^{p-1,p}(x_3,t)$ and $\mathbf{U}^{p,p-1}(x_3,t)$

In this subsection we suppose that $U_i^{p,p}(x_3,t)$, $i = 1,2,3$ were found by (57). Differentiating (44) and (3) $p-1$ times with respect to x_1 and p times with respect to x_2 and using the formula (36) we obtain the system

$$\rho\frac{\partial^2\mathbf{U}^{p-1,p}}{\partial t^2} = \mathbf{B}\frac{\partial^2\mathbf{U}^{p-1,p}}{\partial x_3^2} + p\mathbf{D}\frac{\partial\mathbf{U}^{p,p}}{\partial x_3}, \quad (x_3,t)\in\Delta(P), \tag{58}$$

with initial conditions

$$\mathbf{U}^{p-1,p}\big|_{t=0} = \boldsymbol{\varphi}^{p-1,p}(x_3), \quad \frac{\partial\mathbf{U}^{p-1,p}}{\partial t}\bigg|_{t=0} = \boldsymbol{\psi}^{p-1,p}(x_3), \;\; x_3\in L(P), \tag{59}$$

where $\boldsymbol{\varphi}^{p-1,p}(x_3)$, $\boldsymbol{\psi}^{p-1,p}(x_3)$ are given coefficients of (33) and (34). Differentiating (44) and (3) p times with respect to x_1 and $p-1$ times with respect to x_2 we find

$$\rho\frac{\partial^2\mathbf{U}^{p,p-1}}{\partial t^2} = \mathbf{B}\frac{\partial^2\mathbf{U}^{p,p-1}}{\partial x_3^2} + p\mathbf{E}\frac{\partial\mathbf{U}^{p,p}}{\partial x_3}, \quad (x_3,t)\in\Delta(P), \tag{60}$$

$$\mathbf{U}^{p,p-1}\big|_{t=0} = \boldsymbol{\varphi}^{p,p-1}(x_3), \quad \frac{\partial\mathbf{U}^{p,p-1}}{\partial t}\bigg|_{t=0} = \boldsymbol{\psi}^{p,p-1}(x_3), \quad x_3\in L(P), \tag{61}$$

where $\boldsymbol{\varphi}^{p,p-1}(x_3)$, $\boldsymbol{\psi}^{p,p-1}(x_3)$ are given coefficients of (33) and (34).

Applying the diagonalization process which was described in Section 5.1 and setting

$$\mathbf{U}^{p-1,p} = \mathbf{Z}\mathbf{Y}^{p-1,p}, \tag{62}$$

$$\mathbf{U}^{p,p-1} = \mathbf{Z}\mathbf{Y}^{p,p-1} \tag{63}$$

we find

$$\rho\frac{\partial^2\mathbf{Y}^{p-1,p}}{\partial t^2} = \boldsymbol{\Lambda}\frac{\partial^2\mathbf{Y}^{p-1,p}}{\partial x_3^2} + p\tilde{\mathbf{D}}\frac{\partial\mathbf{Y}^{p,p}}{\partial x_3}, \quad (x_3,t)\in\Delta(P), \tag{64}$$

$$\mathbf{Y}^{p-1,p}\big|_{t=0} = \mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-1,p}(x_3), \quad x_3\in L(P), \tag{65}$$

$$\frac{\partial\mathbf{Y}^{p-1,p}}{\partial t}\bigg|_{t=0} = \mathbf{Z}^{-1}\boldsymbol{\psi}^{p-1,p}(x_3), \quad x_3\in L(P), \tag{66}$$

and

$$\rho\frac{\partial^2\mathbf{Y}^{p,p-1}}{\partial t^2} = \boldsymbol{\Lambda}\frac{\partial^2\mathbf{Y}^{p,p-1}}{\partial x_3^2} + p\tilde{\mathbf{E}}\frac{\partial\mathbf{Y}^{p,p}}{\partial x_3}, \quad (x_3,t)\in\Delta(P), \tag{67}$$

$$\mathbf{Y}^{p,p-1}\big|_{t=0} = \mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p-1}(x_3), \quad x_3\in L(P), \tag{68}$$

$$\frac{\partial\mathbf{Y}^{p,p-1}}{\partial t}\bigg|_{t=0} = \mathbf{Z}^{-1}\boldsymbol{\psi}^{p,p-1}(x_3), \quad x_3\in L(P), \tag{69}$$

from (58), (59) and (60), (61), respectively. Here

$$\tilde{\mathbf{D}} = \mathbf{Z}^{-1}\mathbf{D}\mathbf{Z}, \quad \tilde{\mathbf{E}} = \mathbf{Z}^{-1}\mathbf{E}\mathbf{Z}. \tag{70}$$

Using d'Alembert's formula for (64)–(66) and (67)–(69) and (62), (63) we find for $(x_3,t)\in\Delta(P)$ the following formulas:

$$U_i^{p-1,p}(x_3,t)=\sum_{j=1}^{3}Z_{ij}\left\{\frac{p}{2\rho v_j}\int_0^t\int_{x_3-v_j(t-\tau)}^{x_3+v_j(t-\tau)}\left(\tilde{\mathbf{D}}\left[\mathbf{Z}^{-1}\frac{\partial\mathbf{U}^{p,p}}{\partial x_3}\right]\right)_j(\sigma,\tau)\mathrm{d}\sigma\,\mathrm{d}\tau+\frac{1}{2}\left[(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-1,p})_j(x_3-v_jt)+(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-1,p})_j(x_3+v_jt)\right]+\frac{1}{2v_j}\int_{x_3-v_jt}^{x_3+v_jt}(\mathbf{Z}^{-1}\boldsymbol{\psi}^{p-1,p})_j(\sigma)\mathrm{d}\sigma\right\},\quad i=1,2,3,\ (x_3,t)\in\Delta(P),\tag{71}$$

$$U_i^{p,p-1}(x_3,t)=\sum_{j=1}^{3}Z_{ij}\left\{\frac{p}{2\rho v_j}\int_0^t\int_{x_3-v_j(t-\tau)}^{x_3+v_j(t-\tau)}\left(\tilde{\mathbf{E}}\left[\mathbf{Z}^{-1}\frac{\partial\mathbf{U}^{p,p}}{\partial x_3}\right]\right)_j(\sigma,\tau)\mathrm{d}\sigma\,\mathrm{d}\tau+\frac{1}{2}\left[(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p-1})_j(x_3-v_jt)+(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p-1})_j(x_3+v_jt)\right]+\frac{1}{2v_j}\int_{x_3-v_jt}^{x_3+v_jt}(\mathbf{Z}^{-1}\boldsymbol{\psi}^{p,p-1})_j(\sigma)\mathrm{d}\sigma\right\},\quad i=1,2,3,\ (x_3,t)\in\Delta(P).\tag{72}$$

These are formulas for finding $U_i^{p-1,p}(x_3,t)$, $U_i^{p,p-1}(x_3,t)$, $i=1,2,3$ via $U_j^{p,p}$ (x_3,t), $(x_3,t)\in\Delta(P)$.

5.3. Finding $U^{s,k}(x_3,t)$, $s,k=0,1,\dots,p-1$

5.3.1. Case $s=p-1$, $k=p-1$

Applying the operator $\frac{\partial^{2p-2}}{\partial x_1^{p-1}\partial x_2^{p-1}}$ to (44) and (3) and using (33), (34) and (36) we find

$$\rho\frac{\partial^2\mathbf{U}^{p-1,p-1}}{\partial t^2}=\mathbf{B}\frac{\partial^2\mathbf{U}^{p-1,p-1}}{\partial x_3^2}+p\mathbf{D}\frac{\partial\mathbf{U}^{p,p-1}}{\partial x_3}+p\mathbf{E}\frac{\partial\mathbf{U}^{p-1,p}}{\partial x_3}+p^2\mathbf{F}\mathbf{U}^{p,p},\quad (x_3,t)\in\Delta(P),\tag{73}$$

$$\mathbf{U}^{p-1,p-1}\big|_{t=0}=\boldsymbol{\varphi}^{p-1,p-1}(x_3),\quad x_3\in L(P),\tag{74}$$

$$\left.\frac{\partial\mathbf{U}^{p-1,p-1}}{\partial t}\right|_{t=0}=\boldsymbol{\psi}^{p-1,p-1}(x_3),\quad x_3\in L(P),\tag{75}$$

where $\boldsymbol{\varphi}^{p-1,p-1}(x_3)$, $\boldsymbol{\psi}^{p-1,p-1}(x_3)$ are given coefficients of (33) and (34), **B**, **D**, **E**, **F** are given by (45)–(47).

The solution of (73)–(75) is given by

$$U_i^{p-1,p-1}=\sum_{j=1}^{3}Z_{ij}\left\{\frac{1}{2\rho v_j}\int_0^t\int_{x_3-v_j(t-\tau)}^{x_3+v_j(t-\tau)}\left(p\tilde{\mathbf{D}}\left[\mathbf{Z}^{-1}\frac{\partial\mathbf{U}^{p,p-1}}{\partial x_3}\right]+p\tilde{\mathbf{E}}\left[\mathbf{Z}^{-1}\frac{\partial\mathbf{U}^{p-1,p}}{\partial x_3}\right]+p^2\tilde{\mathbf{F}}\left[\mathbf{Z}^{-1}\mathbf{U}^{p,p}\right]\right)_j(\sigma,\tau)\mathrm{d}\sigma\,\mathrm{d}\tau+\frac{1}{2}\left[(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-1,p-1})_j(x_3-v_jt)+(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-1,p-1})_j(x_3+v_jt)\right]+\frac{1}{2v_j}\int_{x_3-v_jt}^{x_3+v_jt}(\mathbf{Z}^{-1}\boldsymbol{\psi}^{p-1,p-1})_j(\sigma)\mathrm{d}\sigma\right\},\ i=1,2,3,\ (x_3,t)\in\Delta(P),\tag{76}$$

where $\tilde{\mathbf{D}}$, $\tilde{\mathbf{E}}$ are defined by (70) and

$$\tilde{\mathbf{F}}=\mathbf{Z}^{-1}\mathbf{F}\mathbf{Z}.\tag{77}$$

5.3.2. Cases $k = p$, $s = p - m$, $m = 2, \ldots, p$

Applying the operator $\frac{\partial^{2p-m}}{\partial x_1^{p-m} \partial x_2^p}$ to (44) and (3) for $m = 2, \ldots, p$ successively and using (33), (34) and (36) we find

$$\rho \frac{\partial^2 \mathbf{U}^{p-m,p}}{\partial t^2} = \mathbf{B} \frac{\partial^2 \mathbf{U}^{p-m,p}}{\partial x_3^2} + (p-m+1)\mathbf{D} \frac{\partial \mathbf{U}^{p-m+1,p}}{\partial x_3} + (p-m+1)(p-m+2)\mathbf{G}\mathbf{U}^{p-m+2,p},$$

$$(x_3, t) \in \Delta(P), \tag{78}$$

$$\mathbf{U}^{p-m,p}|_{t=0} = \boldsymbol{\varphi}^{p-m,p}(x_3), \quad x_3 \in L(P), \tag{79}$$

$$\left.\frac{\partial \mathbf{U}^{p-m,p}}{\partial t}\right|_{t=0} = \boldsymbol{\psi}^{p-m,p}(x_3), \quad x_3 \in L(P), \tag{80}$$

where $\boldsymbol{\varphi}^{p-m,p}(x_3)$, $\boldsymbol{\psi}^{p-m,p}(x_3)$ are given coefficients of (33) and (34); $\mathbf{B}$, $\mathbf{D}$, $\mathbf{G}$ are given by (45)–(47).

The solutions of (78)–(80) are given by formulas

$$U_i^{p-m,p}(x_3,t) = \sum_{j=1}^{3} Z_{ij} \left\{ \frac{(p-m+1)}{2\rho v_j} \int_0^t \int_{x_3 - v_j(t-\tau)}^{x_3 + v_j(t-\tau)} \left(\tilde{\mathbf{D}} \left[\mathbf{Z}^{-1} \frac{\partial \mathbf{U}^{p-m+1,p}}{\partial x_3} \right] \right.\right.$$
$$\left. + (p-m+2)\tilde{\mathbf{G}}\left[\mathbf{Z}^{-1}\mathbf{U}^{p-m+2,p}\right] \right)_j (\sigma,\tau)\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{1}{2}\left[(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-m,p})_j(x_3 - v_j t) + (\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-m,p})_j(x_3 + v_j t) \right]$$
$$\left. + \frac{1}{2v_j} \int_{x_3 - v_j t}^{x_3 + v_j t} (\mathbf{Z}^{-1}\boldsymbol{\psi}^{p-m,p})_j(\sigma)\mathrm{d}\sigma \right\}, \quad i = 1,2,3, \ (x_3,t) \in \Delta(P), \ m = 2, \ldots, p, \tag{81}$$

where $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{F}}$ are given by (70) and (77), and

$$\tilde{\mathbf{G}} = \mathbf{Z}^{-1}\mathbf{G}\mathbf{Z}. \tag{82}$$

5.3.3. Cases $s = p$, $k = p - m$, $m = 2, \ldots, p$

Taking the derivative of (44) and (3) p times with respect to x_1, $p-m$ times with respect to x_2 and using (33), (34) and (36) we get

$$\rho \frac{\partial^2 \mathbf{U}^{p,p-m}}{\partial t^2} = \mathbf{B} \frac{\partial^2 \mathbf{U}^{p,p-m}}{\partial x_3^2} + (p-m+1)\mathbf{E} \frac{\partial \mathbf{U}^{p,p-m+1}}{\partial x_3} + (p-m+1)(p-m+2)\mathbf{H}\mathbf{U}^{p,p-m+2},$$

$$(x_3, t) \in \Delta(P), \tag{83}$$

$$\mathbf{U}^{p,p-m}|_{t=0} = \boldsymbol{\varphi}^{p,p-m}(x_3), \quad x_3 \in L(P), \tag{84}$$

$$\left.\frac{\partial \mathbf{U}^{p,p-m}}{\partial t}\right|_{t=0} = \boldsymbol{\psi}^{p,p-m}(x_3), \quad x_3 \in L(P), \tag{85}$$

where $\boldsymbol{\varphi}^{p,p-m}(x_3)$, $\boldsymbol{\psi}^{p,p-m}(x_3)$ are given coefficients of (33) and (34); $\mathbf{B}$, $\mathbf{E}$, $\mathbf{H}$ are given by (46) and (47).

The solution of (83)–(85) is given by

$$U_i^{p,p-m}(x_3,t) = \sum_{j=1}^{3} Z_{ij} \left\{ \frac{(p-m+1)}{2\rho v_j} \int_0^t \int_{x_3 - v_j(t-\tau)}^{x_3 + v_j(t-\tau)} \left(\tilde{\mathbf{E}} \left[\mathbf{Z}^{-1} \frac{\partial \mathbf{U}^{p,p-m+1}}{\partial x_3} \right] \right.\right.$$
$$\left. + (p-m+2)\tilde{\mathbf{H}}\left[\mathbf{Z}^{-1}\mathbf{U}^{p,p-m+2}\right] \right)_j (\sigma,\tau)\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{1}{2}[(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p-m})_j(x_3 - v_j t)$$
$$\left. + (\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p,p-m})_j(x_3 + v_j t)] + \frac{1}{2v_j} \int_{x_3 - v_j t}^{x_3 + v_j t} (\mathbf{Z}^{-1}\boldsymbol{\psi}^{p,p-m})_j(\sigma)\mathrm{d}\sigma \right\},$$

$$i = 1,2,3, \quad (x_3,t) \in \Delta(P), \quad m = 2, \ldots, p, \tag{86}$$

where $\tilde{\mathbf{E}}$ is given by (70) and

$$\tilde{\mathbf{H}} = \mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}. \tag{87}$$

5.3.4. Cases $s = p - m$, $k = p - n$, $m, n = 2,\ldots,p$

Taking the derivative of (44) and (3) $p - m$ times with respect to x_1, $p - n$ times with respect to x_2 and using (33), (34) and (36) we get

$$\begin{aligned}\rho\frac{\partial^2\mathbf{U}^{p-m,p-n}}{\partial t^2} &= \mathbf{B}\frac{\partial^2\mathbf{U}^{p-m,p-n}}{\partial x_3^2} + (p-m+1)\mathbf{D}\frac{\partial\mathbf{U}^{p-m+1,p-n}}{\partial x_3} + (p-n+1)\mathbf{E}\frac{\partial\mathbf{U}^{p-m,p-n+1}}{\partial x_3}\\ &\quad + (p-m+1)(p-n+1)\mathbf{F}\mathbf{U}^{p-m+1,p-n+1} + (p-m+1)(p-m+2)\mathbf{G}\mathbf{U}^{p-m+2,p-n}\\ &\quad + (p-n+1)(p-n+2)\mathbf{H}\mathbf{U}^{p-m,p-n+2}, (x_3,t) \in \Delta(P),\end{aligned} \tag{88}$$

$$\mathbf{U}^{p-m,p-n}\big|_{t=0} = \boldsymbol{\varphi}^{p-m,p-n}(x_3), \quad x_3 \in L(P), \tag{89}$$

$$\left.\frac{\partial\mathbf{U}^{p-m,p-n}}{\partial t}\right|_{t=0} = \boldsymbol{\psi}^{p-m,p-n}(x_3), \quad x_3 \in L(P), \tag{90}$$

where $\boldsymbol{\varphi}^{p-m,p-n}(x_3)$, $\boldsymbol{\psi}^{p-m,p-n}(x_3)$ are given coefficients of (33) and (34); $\mathbf{B}$, $\mathbf{D}$, $\mathbf{G}$, $\mathbf{E}$, $\mathbf{F}$, $\mathbf{H}$ are given by (45)–(47).

The solution of (88)–(90) is given by

$$\begin{aligned}U_i^{p-m,p-n}(x_3,t) = \sum_{j=1}^{3} Z_{ij}\Bigg\{&\frac{1}{2\rho v_j}\int_0^t\int_{x_3-v_j(t-\tau)}^{x_3+v_j(t-\tau)}\Bigg((p-m+1)\tilde{\mathbf{D}}\left[\mathbf{Z}^{-1}\frac{\partial\mathbf{U}^{p-m+1,p-n}}{\partial x_3}\right]\\ &+(p-n+1)\tilde{\mathbf{E}}\left[\mathbf{Z}^{-1}\frac{\partial\mathbf{U}^{p-m,p-n+1}}{\partial x_3}\right] + (p-m+1)(p-n+1)\tilde{\mathbf{F}}\left[\mathbf{Z}^{-1}\mathbf{U}^{p-m+1,p-n+1}\right]\\ &+(p-m+1)(p-m+2)\tilde{\mathbf{G}}\left[\mathbf{Z}^{-1}\mathbf{U}^{p-m+2,p-n}\right] + (p-n+1)(p-n+2)\tilde{\mathbf{H}}\left[\mathbf{Z}^{-1}\mathbf{U}^{p-m,p-n+2}\right]\Bigg)_j(\sigma,\tau)\mathrm{d}\sigma\mathrm{d}\tau\\ &+\frac{1}{2}\left[(\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-m,p-n})_j(x_3-v_jt) + (\mathbf{Z}^{-1}\boldsymbol{\varphi}^{p-m,p-n})_j(x_3+v_jt)\right]\\ &+\frac{1}{2v_j}\int_{x_3-v_jt}^{x_3+v_jt}(\mathbf{Z}^{-1}\boldsymbol{\psi}^{p-m,p-n})_j(\sigma)\mathrm{d}\sigma\Bigg\}, \; i=1,2,3, \; (x_3,t)\in\Delta(P), \; m,n=2,\ldots,p,\end{aligned} \tag{91}$$

where $\tilde{\mathbf{D}}$, $\tilde{\mathbf{E}}$, $\tilde{\mathbf{F}}$, $\tilde{\mathbf{G}}$, $\tilde{\mathbf{H}}$ are given before.

6. PS-method correctness and simulation examples

In this section we present numerical examples that demonstrate the correctness and efficiency of the proposed approach. We consider an example of the Cauchy problem for the isotropic elastic system to show the correctness of PS-method described in the previous section. The reason to use here the isotropic medium is the following. We know that the solution of the Cauchy problem only for isotropic elasticity may be found by a reduction of the initial value problem of elasticity to the initial value problems of wave equations for scalar and vector elastic potentials (Tikhonov and Samarskii, 1963). This is well known classical method which we will call SVP (scalar and vector potentials) method. SVP-method is completely different from PS-method. The correctness of the proposed approach is established by the comparison of solutions found by these two different methods.

In this section there are several examples of numerical solutions of initial value problems for different cases of anisotropy for the same initial data. These examples show the robustness of the proposed method for the simulation of wave propagation in anisotropic media.

6.1. Test of PS-method correctness

Let us construct a solution of the Cauchy problem for an isotropic elastic system using the wave equations for the scalar and vector elastic potentials.

Let $\rho = 2.203$ and matrix $\mathbf{C} = (C_{mn})_{6\times 6}$ be the matrix with the following elements:

$$C_{11} = C_{22} = C_{33} = \lambda + 2\mu, \quad C_{12} = C_{23} = C_{13} = \lambda, \quad C_{44} = C_{55} = C_{66} = \mu, \quad \lambda = 1.61, \quad \mu = 3.12.$$

Other elements of the matrix $\mathbf{C}$ are equal to zero. Let us consider the scalar functions $g(x)$, $h(x)$ and vector functions $\mathbf{g}(x)$, $\mathbf{h}(x)$ which are defined as follows:

$$\begin{aligned} &g(x) = x_1^3 x_2^3 \sin x_3 + x_1 x_2 \cos x_3, \quad h(x) = x_1^3 x_2^3 \sin x_3 + x_1 x_2 \sin x_3, \\ &\mathbf{g}(x) = (g_1, g_2, g_3), \quad \mathbf{h}(x) = (h_1, h_2, h_3), \\ &g_1(x) = x_1^6 x_2^6 \sin x_3, \quad h_1(x) = x_1^5 x_2^5 \sin x_3, \\ &g_2(x) = x_1^6 x_2^5 \cos x_3, \quad h_2(x) = x_1^5 x_2^4 \cos x_3, \\ &g_3(x) = x_1^5 x_2^5 \sin x_3, \quad h_3(x) = x_1^6 x_2^5 \sin x_3. \end{aligned} \tag{92}$$

Applying operators of gradient ∇_x to $g(x)$ and $h(x)$, and curl_x to $\mathbf{g}(x)$ and $\mathbf{h}(x)$, we find explicitly the following vector functions

$$\boldsymbol{\varphi}(x) = \nabla_x g(x) + \mathrm{curl}_x \mathbf{g}(x), \tag{93}$$

$$\boldsymbol{\psi}(x) = \nabla_x h(x) + \mathrm{curl}_x \mathbf{h}(x). \tag{94}$$

The 3-D images of second components of $\boldsymbol{\varphi}(x)$ and $\boldsymbol{\psi}(x)$ for $x_2 = 10$ are shown on the left- and right-hand sides of Fig. 1.

Let the function $v(x,t)$ be a solution of the following Cauchy problem for the scalar wave equation

$$\rho \frac{\partial^2 v(x,t)}{\partial t^2} = (\lambda + 2\mu)\Delta_x v(x,t), \quad x \in \mathbb{R}^3, \ t > 0, \tag{95}$$

$$v|_{t=0} = g(x), \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = h(x), \ \ x \in \mathbb{R}^3, \tag{96}$$

and the vector-function $\mathbf{w}(x,t)$ be a solution of the following Cauchy problem for the vector wave equation

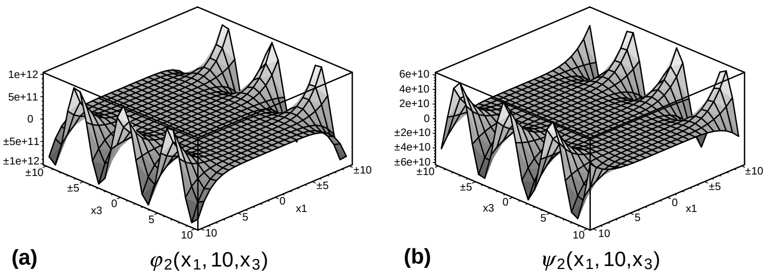

Fig. 1. The second components of initial vector functions: (a) $\varphi_2(x_1, 10, x_3)$ and (b) $\psi_2(x_1, 10, x_3)$.

$$\rho \frac{\partial^2 \mathbf{w}(x,t)}{\partial t^2} = \mu \Delta_x \mathbf{w}(x,t), \quad x \in \mathbb{R}^3, \ t > 0, \tag{97}$$

$$\mathbf{w}|_{t=0} = \mathbf{g}(x), \quad \left.\frac{\partial \mathbf{w}}{\partial t}\right|_{t=0} = \mathbf{h}(x), \ x \in \mathbb{R}^3. \tag{98}$$

Then the vector-function

$$\mathbf{u}(x,t) = \nabla_x v(x,t) + \mathrm{curl}_x \mathbf{w}(x,t) \tag{99}$$

will be a solution of the Cauchy problem for the following system of isotropic elasticity:

$$\rho \frac{\partial^2 \mathbf{u}(x,t)}{\partial t^2} = (\lambda + 2\mu)\nabla_x \mathrm{div}_x \mathbf{u}(x,t) - \mu \mathrm{curl}_x \mathrm{curl}_x \mathbf{u}(x,t), \quad x \in \mathbb{R}^3, \ t > 0, \tag{100}$$

$$\mathbf{u}|_{t=0} = \boldsymbol{\varphi}(x), \quad \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \boldsymbol{\psi}(x), \ x \in \mathbb{R}^3, \tag{101}$$

where $\boldsymbol{\varphi}(x)$, $\boldsymbol{\psi}(x)$ are defined by (93) and (94).

The Cauchy problem (100) and (101) is the main object of this subsection. The solution of (100) and (101) was found numerically by the formula (99) in which $\nabla_x v(x,t)$ and $\mathrm{curl}_x \mathbf{w}(x,t)$ were defined by the following rules. The solution $v(x,t)$ of (95) and (96) we find in the following form by Kirchhoff's formula and spherical coordinates

$$v(x,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial}{\partial t}(tg(x+at\mathbf{v})) + th(x+at\mathbf{v})\right]\mathrm{d}\omega_{\mathbf{v}}, \tag{102}$$

where

$$\begin{aligned} &a^2 = \frac{\lambda+2\mu}{\rho}, \quad \mathbf{v} = (v_1, v_2, v_3), \\ &v_1 = \cos\gamma\sin\theta, \quad v_2 = \sin\gamma\sin\theta, \quad v_3 = \cos\theta, \\ &0 \leqslant \gamma < 2\pi, \quad 0 < \theta < \pi, \quad \mathrm{d}\omega_{\mathbf{v}} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\gamma. \end{aligned} \tag{103}$$

Using (102) we find $\nabla_x v(x,t)$ as follows:

$$\nabla_x v(x,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\nabla_x\left[\frac{\partial}{\partial t}(tg(x+at\mathbf{v})) + th(x+at\mathbf{v})\right]\mathrm{d}\omega_{\mathbf{v}}. \tag{104}$$

We note that the partial derivatives of integrands and integrals with respect to ψ were found analytically using Maple 7, and then integrals with respect to θ were calculated using the trapezoid rule. Applying curl_x to the Kirchhoff's formula for the solution $\mathbf{w}(x,t)$ of (97) and (98) we find

$$\mathrm{curl}_x \mathbf{w}(x,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\mathrm{curl}_x\left[\frac{\partial}{\partial t}(t\mathbf{g}(x+at\mathbf{v})) + t\mathbf{h}(x+at\mathbf{v})\right]\mathrm{d}\omega_{\mathbf{v}}. \tag{105}$$

The calculation of (105) is similar to (104), the partial derivatives of integrands are found analytically using Maple 7, and then integrals are calculated. From the other hand the solution of (100) and (101) can be found numerically using PS-method. As a result of it we have two different numerical methods for the solution of the same Cauchy problem (100) and (101). 3-D images of $u_2(x,t)$ for fixed x_2 and different values of time variable are shown in Fig. 2. Here the horizontal axes are x_1 and x_3, the vertical axis is u_2-axis for $x_2 = 10$ and $t = 2$, $t = 15$, $t = 20$, respectively. The left-hand side column of images is obtained by SVP-method, the right-hand side column corresponds to PS-method. Tables 1–3 contain numerical results in numbers for $\hat{u}_2(3,10,x_3,t) = u_2(3,10,x_3,t) \times 10^{-11}$ found by SVP-method and PS-method for $x_3 = -10$, $x_3 = -5$, $x_3 = 0$, $x_3 = 5$, $x_3 = 10$ and different values of t: $t = 2$ (Table 1), $t = 15$ (Table 2), $t = 20$ (Table 3).

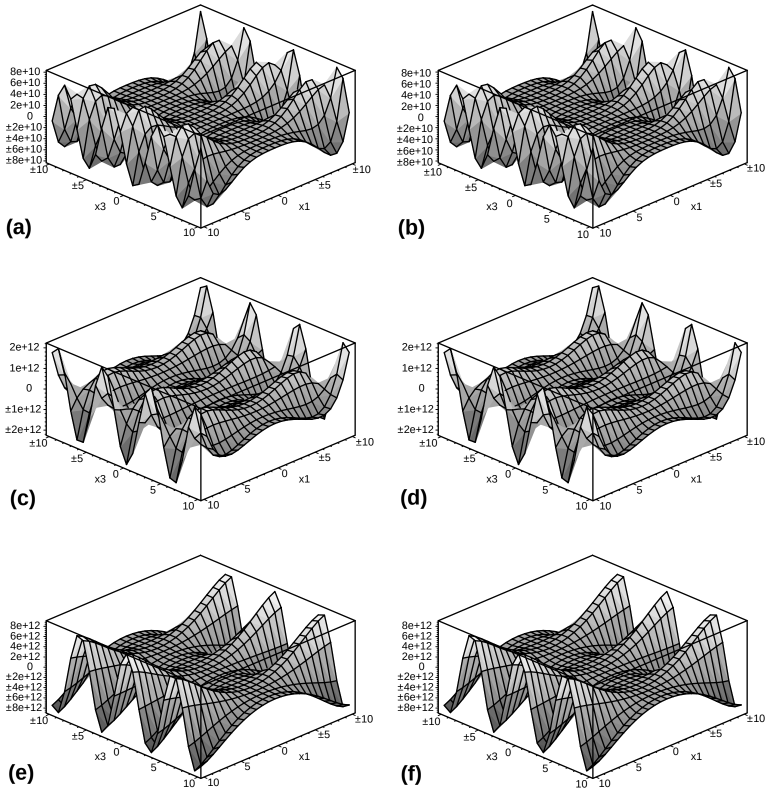

Fig. 2. 3-D images of $u_2(x_1, 10, x_3, t)$ found by SVP and PS methods for $t = 2$, $t = 15$, $t = 20$: (a) SVP-method, $t = 2$, (b) PS-method, $t = 2$, (c) SVP-method, $t = 15$, (d) PS-method, $t = 15$, (e) SVP-method, $t = 20$ and (f) PS-method, $t = 20$.

The similar images and tables we can find for other two components $u_1(x, t)$, $u_3(x, t)$ and different values x, t. We confirm the correctness of PS-method by the comparison of these images and numerical results.

6.2. Simulation examples of wave propagations in anisotropic media by PS-method

The initial value problem (2) and (3) for several cases of $\mathbf{C}$, ρ and the same initial functions is solved numerically. The visualization of numerical solutions of these problems are presented in figures below. The initial functions here are given by the following relations:

$$\begin{aligned} &\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3), \quad \boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3), \\ &\varphi_j = p(x_1)p(x_2)p(x_3), \quad \psi_j \equiv 0, \;\; j = 1, 2, 3, \end{aligned} \tag{106}$$

where

$$\begin{aligned} p(z) = {} & 2.49961 - 2.59991z^2 + 0.806848z^4 - 0.117057z^6 + 0.00941347z^8 - 0.000439476z^{10} \\ & + 0.0000112076z^{12} - 1.20324 \times 10^{-7}z^{14}. \end{aligned}$$

The function $p(z)$ here was found by Mathematica 4 as an interpolating polynomial of

$$f(z) = \frac{1}{z}\sin\left(\frac{5z}{2}\right) \tag{107}$$

in the interval $[-5, 5]$ with 16 points.

Table 1
Numerical comparison of SVP and PS methods ($t = 2$)

x_3	$\hat{u}_2$ (SVP)	$\hat{u}_2$ (PS)	$\lvert\hat{u}_2(\text{SVP}) - \hat{u}_2(\text{PS})\rvert$
−10	−0.05209188934	−0.05209155134	0.000000338
−5	0.01053475910	0.01053471708	0.00000004202
0	0.05806851491	0.05806815303	0.00000036188
5	0.02240892456	0.02240876131	0.00000016325
10	−0.04535538582	−0.04535511670	0.00000026912

Table 2
Numerical comparison of SVP and PS methods ($t = 15$)

x_3	$\hat{u}_2$ (SVP)	$\hat{u}_2$ (PS)	$\lvert\hat{u}_2(\text{SVP}) - \hat{u}_2(\text{PS})\rvert$
−10	−4.128852746	−4.128855306	0.00000256
−5	1.208141473	1.208142533	0.00000106
0	4.814261476	4.814264009	0.000002533
5	1.523105374	1.523106767	0.000001393
10	−3.950166243	−3.950168420	0.000002177

Table 3
Numerical comparison of SVP and PS methods ($t = 20$)

x_3	$\hat{u}_2$ (SVP)	$\hat{u}_2$ (PS)	$\lvert\hat{u}_2(\text{SVP}) - \hat{u}_2(\text{PS})\rvert$
−10	−8.955222938	−8.955253915	0.000030977
−5	2.721805168	2.721813295	0.000008127
0	10.49937635	10.49940493	0.00002858
5	3.234737115	3.234755002	0.000017887
10	−8.664219403	−8.664249582	0.000030179

Let $(x_1, x_2, x_3) \in \mathbb{R}^3$ be space variables and one of these variables be fixed, for example $x_2 = 0$. The three-dimensional graph of each function $\varphi_j(x)$ has a hillock shape which is shown in Fig. 3. The horizontal axes here are x_1, x_3, the vertical axis is φ_j for $x_2 = 0$. In Fig. 3 level plots of the same surface are shown. The different colors correspond to different levels of the surface.

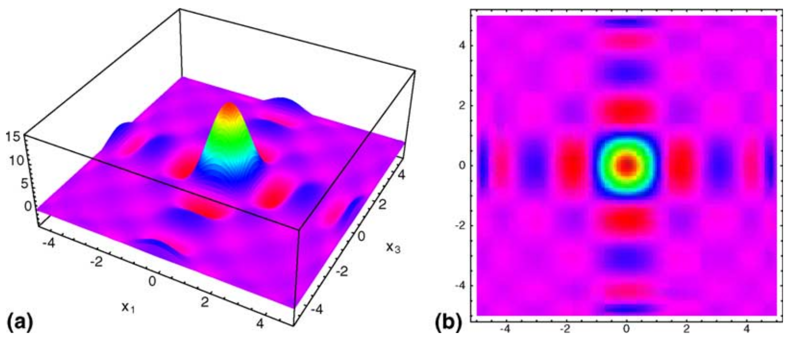

Fig. 3. The third component of initial vector function: (a) 3-D level plot of $\varphi_j(x_1, 0, x_3)$ and (b) 2-D level plot of $\varphi_j(x_1, 0, x_3)$.

Let us consider now two Cauchy problems (2) and (3) with the same initial vector-functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, which are given by (106), for the following two cases of the matrix $\mathbf{C}$ and ρ.

Case 1 corresponds to $\rho = 4.64$ and the matrix $\mathbf{C} = (C_{mn})_{6\times 6}$ elements of which are defined as

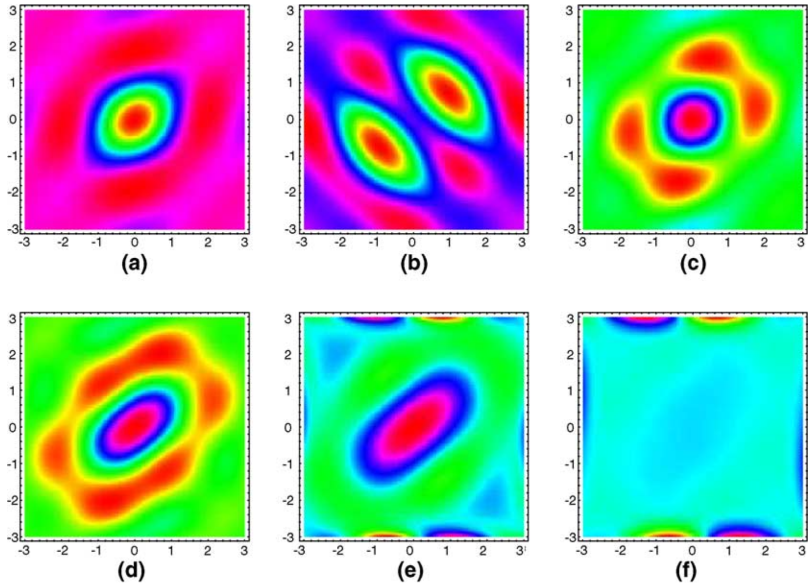

Fig. 4. $u_1(x,t)$ for orthorhombic media: (a) $t = 0.5$, (b) $t = 1$ (c), $t = 1.5$, (d) $t = 2$, (e) $t = 2.5$ and (f) $t = 3$.

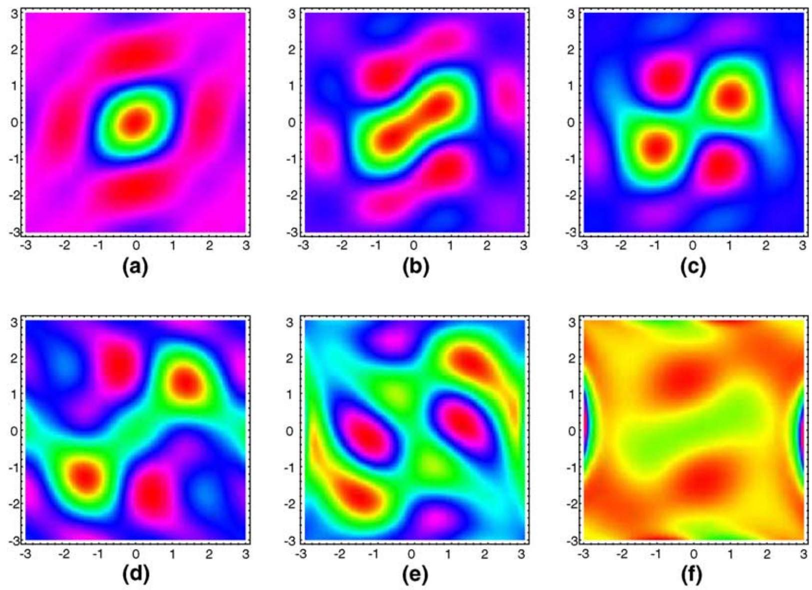

Fig. 5. $u_1(x,t)$ for tetragonal media: (a) $t = 0.5$, (b) $t = 1$, (c) $t = 1.5$, (d) $t = 2$, (e) $t = 2.5$ and (f) $t = 3$.

$$\begin{aligned} &C_{11} = 3.01, \quad C_{12} = 1.61, \quad C_{13} = 1.11, \\ &C_{22} = 5.8, \quad C_{23} = 0.80, \quad C_{33} = 4.29, \\ &C_{44} = 1.69, \quad C_{55} = 2.06, \quad C_{66} = 1.58, \end{aligned} \tag{108}$$

other elements are equal to zero. Case 1 is relative to materials with the orthorhombic structure.

Case 2 is given by $\rho = 7.28$ and the matrix $\mathbf{C} = (C_{mn})_{6\times 6}$ of the form

$$\begin{aligned} &C_{11} = C_{22} = 4.53, \quad C_{12} = 4.00, \\ &C_{13} = C_{23} = 4.15, \quad C_{33} = 4.51, \\ &C_{44} = C_{55} = 0.651, \quad C_{66} = 1.21, \end{aligned} \tag{109}$$

other elements are equal to zero. Case 2 corresponds to the materials with the tetragonal structure.

The solutions of the Cauchy problem (2) and (3) with initial functions (106) for these two cases of $\mathbf{C}$, ρ were found by PS-method numerically. In Cartesian coordinates x_1, x_3 we plot the values of u_1 for $x_2 = 0$ and $t =$ constant as rectangular array of cells with colors on a surface. Different colors correspond to different values of $u_1(x_1,0,x_3,t)$, $t =$ constant (different level of points on the surface). In Figs. 4 and 5 these plots are shown for $t = 0.5$, $t = 1$, $t = 1.5$, $t = 2$, $t = 2.5$, $t = 3$.

Two other components can be represented by the colored images as well. We can see from the pictures how the wave propagations in different anisotropic media depend on the type of anisotropy.

7. Conclusion

In this paper we have considered the initial value problem (IVP) for the linear anisotropic elastic system. The theory, the method of solving this IVP and the simulation of the elastic wave propagation using this method have been studied. The existence and uniqueness theorems for weak and smooth solutions have been proved by the reduction of IVP for the original system to IVP for symmetric hyperbolic system of the first order. All our arguments can be directly generalized for non-homogeneous linear elastic system with smooth function coefficients depending on space and time variables. We have proved the theorem saying that the solution of IVP for the linear anisotropic elastic system has a polynomial form with respect to lateral variables if initial data are polynomials with respect to the same lateral variables.

This theorem gives an opportunity to get a method of the IVP solution for linear anisotropic elastic system. The central point of our paper is the description of this method which we called Polynomial Solution method (PS-method). Using this method we have found a solution of IVP for linear anisotropic elastic system in the conoid of dependence. We note here that the polynomial structure of the solution for IVP with polynomial data and PS-method can be generalized for the non-homogeneous elastic anisotropic system with function coefficients depending on x_3 (vertical) variable only. This generalization was omitted in our paper to make it more simple. We have shown by numerical examples that the PS-method is robust for the simulation of elastic wave propagation in anisotropic media for the case when the initial data are polynomials with respect to lateral variables.

We note that if the initial data contain continuous functions which are not polynomials we can approximate them by polynomials and then PS-method can be applied to find an approximate solution. The correctness to use PS-method for the solution of IVP for linear anisotropic elasticity with non-polynomial initial data is the topic of our further study.

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Appendix A. Matrix theory facts

This appendix contains several classical facts from matrix theory (Goldberg, 1992).

Theorem 6. *Let* $\mathbf{C}$ *be a real symmetric positive definite matrix of the size* $m\times m$, *where* m *is an arbitrary positive integer. Then* $\mathbf{C}^{-1}$ *is a real symmetric positive definite matrix.*

Proof. Since $\mathbf{C}^{-1}\mathbf{C} = \mathbf{C}\mathbf{C}^{-1} = \mathbf{I}$, using the symmetry property of $\mathbf{C}$ and the rule $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$ we get $\mathbf{I} = \mathbf{C}(\mathbf{C}^{-1})^*$. Multiplying both sides of the last equality by $\mathbf{C}^{-1}$ from left-hand side we get $\mathbf{C}^{-1} = (\mathbf{C}^{-1})^*$ which implies symmetry property of $\mathbf{C}^{-1}$. A matrix is positive definite if and only if its eigenvalues are positive. Using this fact we find that $\mathbf{C}^{-1}$ is positive definite. □

Theorem 7. *Let* $\mathbf{C}$ *be a real symmetric positive definite matrix of the size* $m\times m$, *where* m *is an arbitrary positive integer. Then there exists a real symmetric positive definite matrix* $\mathbf{M}$ *such that* $\mathbf{C}^{-1} = \mathbf{M}^2$.

Proof. According to Theorem 6, $\mathbf{C}^{-1}$ is real symmetric positive definite and is congruent to a diagonal matrix of its eigenvalues. That is, there exists an orthogonal matrix $\mathbf{Q}$ such that

$$\mathbf{Q}^*\mathbf{C}^{-1}\mathbf{Q} = \Lambda, \quad \mathbf{Q}^* = \mathbf{Q}^{-1}. \tag{A.1}$$

Since $\mathbf{C}^{-1}$ is positive definite and symmetric, its eigenvalues λ_i, $i = 1,2,\ldots,m$ are real and nonnegative. Let $\Lambda^{\frac{1}{2}}$ be defined as follows

$$\Lambda^{\frac{1}{2}} = \mathrm{diag}(\lambda_i^{\frac{1}{2}}, i = 1, 2, \ldots, m). \tag{A.2}$$

Now set $\mathbf{M} = \mathbf{Q}\Lambda^{\frac{1}{2}}\mathbf{Q}^*$. Since $\mathbf{Q}$ is orthogonal, $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$, and therefore

$$\mathbf{M}^2 = (\mathbf{Q}\Lambda^{\frac{1}{2}}\mathbf{Q}^*)(\mathbf{Q}\Lambda^{\frac{1}{2}}\mathbf{Q}^*) = \mathbf{Q}\Lambda\mathbf{Q}^* = \mathbf{C}^{-1}. \tag{A.3}$$

Clearly, $\mathbf{M} = \mathbf{Q}\Lambda^{\frac{1}{2}}\mathbf{Q}^*$ is positive definite. □

Theorem 8. *Let* $\mathbf{A}_j$, $\mathbf{S}$ *be real symmetric matrices of the size* $m\times m$, *where* m *is an arbitrary positive integer. Then the matrix* $\tilde{\mathbf{A}}_j = \mathbf{S}\mathbf{A}_j\mathbf{S}$ *is real and symmetric.*

Proof. The proof follows from equalities

$$\tilde{\mathbf{A}}_j^* = (\mathbf{S}\mathbf{A}_j\mathbf{S})^* = \mathbf{S}^*(\mathbf{S}\mathbf{A}_j)^* = \mathbf{S}^*\mathbf{A}_j^*\mathbf{S}^* = \mathbf{S}\mathbf{A}_j\mathbf{S} = \tilde{\mathbf{A}}_j. \qquad \square \tag{A.4}$$

Appendix B. Some existence and uniqueness theorems of symmetric hyperbolic systems theory

This appendix contains results about existence and uniqueness of the Cauchy problem solution for symmetric hyperbolic systems of the first order (Mizohata, 1973). We state here these results in terms and forms which are convenient for us. We use the same notations of the functional spaces which were given before the Theorem 1. Moreover the space $\mathscr{C}([0,T];X)$ and the Sobolev space $\mathscr{H}^k(\mathbb{R}^3)$ are defined as follows. Let X denote a real Banach space with the norm $\|\cdot\|$. Then the space $\mathscr{C}([0,T];X)$ consists of all continuous functions u: $[0,T] \to X$ with

$$\|u\|_{\mathscr{C}([0,T];X)} = \max_{0\leqslant t\leqslant T} \|u(t)\| < \infty. \tag{B.1}$$

The Sobolev space $\mathscr{H}^k(\mathbb{R}^3)$ consists of all locally integrable functions $u : \mathbb{R}^3 \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leqslant k$, $D^\alpha u$ exists in the weak sense and belongs to $\mathscr{L}^2(\mathbb{R}^3)$.

Consider the initial value problem for the symmetric hyperbolic system

$$\frac{\partial \mathbf{V}}{\partial t}+\sum_{j=1}^{3}\tilde{\mathbf{A}}_j\frac{\partial \mathbf{V}}{\partial x_j}=\mathbf{f}(x,t),\quad x\in\mathbb{R}^3,\ t\in(0,T), \tag{B.2}$$

$$\mathbf{V}(x,0)=\mathbf{V}^0,\quad x\in\mathbb{R}^3, \tag{B.3}$$

where T is a fixed positive number, $\mathbf{V}=(V_1,V_2,\ldots,V_9)$ is the vector function with components $V_j=V_j(x,t)$, $j=1,2,\ldots,9$, $\tilde{\mathbf{A}}_j$, $j=1,2,3$ are real symmetric matrices with constant elements of the order 9×9.

The following theorem is the existence and uniqueness theorem of a weak solution of (B.2) and (B.3).

Theorem 9. *Let* $\mathbf{V}^0\in\mathscr{H}^1(\mathbb{R}^3;\mathbb{R}^9)$, $\mathbf{f}\in\mathscr{C}([0,T];\mathscr{H}^1(\mathbb{R}^3;\mathbb{R}^9))$. *Then* (B.2) *and* (B.3) *has a unique solution* $\mathbf{V}(x,t)$ *such that*

$$\mathbf{V}\in\mathscr{C}([0,T];\mathscr{H}^1(\mathbb{R}^3;\mathbb{R}^9))\bigcap\mathscr{C}^1([0,T];\mathscr{L}^2(\mathbb{R}^3;\mathbb{R}^9)). \tag{B.4}$$

The statement and the proof of this theorem can be found in the book Mizohata (1973).

Using the Sobolev lemma (Mizohata, 1973), Theorem 6.4 and corollary (Mizohata, 1973, pp. 335–336) we obtain the existence and uniqueness theorem for genuine solution of (B.2) and (B.3). This has the form:

Theorem 10. *Let* $\mathbf{V}^0\in\mathscr{H}^3(\mathbb{R}^3;\mathbb{R}^9)$, $\mathbf{f}\in\mathscr{C}([0,T];\mathscr{H}^3(\mathbb{R}^3;\mathbb{R}^9))$. *Then the solution of* (B.2) *and* (B.3) *belongs to the class*

$$\mathscr{C}([0,T];\mathscr{C}^1(\mathbb{R}^3;\mathbb{R}^9))\bigcap\mathscr{C}^1([0,T];\mathscr{C}(\mathbb{R}^3;\mathbb{R}^9)). \tag{B.5}$$

Using the multiindex notation (see (24)) and Theorem 10 we find

Theorem 11. *Let* $\gamma=(\gamma_1,\gamma_2,0)$ *be an arbitrary multiindex,* $D^\gamma\mathbf{V}^0\in\mathscr{H}^3(\mathbb{R}^3;\mathbb{R}^9)$, $D^\gamma\mathbf{f}\in\mathscr{C}([0,T];\mathscr{H}^3(\mathbb{R}^3;\mathbb{R}^9))$. *Then the solution of* (B.2), (B.3) *satisfies*

$$D^\gamma\mathbf{V}\in\mathscr{C}([0,T];\mathscr{C}^1(\mathbb{R}^3;\mathbb{R}^9))\bigcap\mathscr{C}^1([0,T];\mathscr{C}(\mathbb{R}^3;\mathbb{R}^9)). \tag{B.6}$$

Appendix C. Domain of dependence and local uniqueness theorem for symmetric hyperbolic systems

In this appendix we describe several facts of the symmetric hyperbolic systems theory (Courant and Hilbert, 1962). These facts are related to the domain of dependence for symmetric hyperbolic systems. We introduce a space-like lens by the domain of dependence and prove the uniqueness theorem inside of this lens.

Let $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ be a space variable, t be a time variable. Consider the symmetric hyperbolic system of the form

$$\frac{\partial \mathbf{u}}{\partial t}+\sum_{j=1}^{3}\mathbf{A}_j\frac{\partial \mathbf{u}}{\partial x_j}=0, \tag{C.1}$$

where $\mathbf{A}_j$, $j=1,2,3$ are symmetric matrices with constant elements. Let P be an arbitrary point with coordinates (x^0,t^0), $t^0>0$; $\Gamma(P)$ be the conoid of the dependence for the symmetric hyperbolic system (C.1) (Courant and Hilbert, 1962); $\Re(h)$ be the surfaces consisting of the plane $t=h$, $(0<h<t^0)$ inside the conoid plus the mantle $\Re^*(h)$ of the conoid $\Gamma(P)$ between $t=0$ and $t=h$; a space-like lens $L(h)$ be defined as interior of the surface $\Re(h)$ for $0\leqslant t\leqslant h$. The boundary $\partial L(h)$ of the lens $L(h)$ consists of the mantle $\Re^*(h)$ and two space-like surfaces $S(0)$ and $S(h)$. The surfaces $S(0)$ and $S(h)$ are the parts of the planes $t=0$ and $t=h$, respectively, inside the conoid $\Gamma(P)$.

There is the following fact for symmetric hyperbolic systems of first order (Courant and Hilbert, 1962) which we state as a proposition.

Proposition 12. *Let* $h \in (0, t^0)$ *and the mantle* $\mathfrak{R}^*(h)$ *of the conoid of the dependence* $\Gamma(P)$ *be defined by the relation* $\Phi(x,t) = 0$, *where* $\Phi(x,t) \in \mathscr{C}^1(\mathbb{R}^3 \times [0,h])$. *Then the characteristic matrix*

$$\mathbf{I}\frac{\partial \Phi(x,t)}{\partial t} + \sum_{j=1}^{3} \mathbf{A}_j \frac{\partial \Phi(x,t)}{\partial x_j} \tag{C.2}$$

is nonnegative on the mantle $\mathfrak{R}^*(h)$.

Further we state and prove a lemma about energy inequalities for the solution of (C.1) inside the lens.

Lemma 13. *Let T be a positive number*, x^0 *be an arbitrary point of* $\mathbb{R}^3$, $P = (x^0, T)$, $\Gamma(P)$ *be the conoid of the dependence for symmetric hyperbolic system* (C.1), $\mathbf{u}(x,t) \in \mathscr{C}([0,T]; \mathscr{H}^1(\mathbb{R}^3;\mathbb{R}^3)) \bigcap \mathscr{C}^1([0,T]; \mathscr{L}^2(\mathbb{R}^3;\mathbb{R}^3))$ *be a solution of* (C.1). *Then the following energy inequality states*:

$$\|\mathbf{u}\|_{S(h)} \leqslant \|\mathbf{u}\|_{S(0)}, \tag{C.3}$$

where $S(h)$ *and* $S(0)$ *are the parts of the planes* $t = h$ *and* $t = 0$, *respectively, inside the conoid* $\Gamma(P)$, $h \in (0, T)$;

$$\|\mathbf{u}\|^2_{S(h)} = \sum_{j=1}^{3} \int_{S(h)} |u_j(x,h)|^2 \, \mathrm{d}x. \tag{C.4}$$

Proof. Consider the conoid of the dependence $\Gamma(P)$ and the lens $L(h)$. The boundary $\partial L(h)$ of the lens may be presented as $\partial L(h) = S(0) \cup S(h) \cup \mathfrak{R}^*(h)$, where $\mathfrak{R}^*(h)$ is the mantle of $\Gamma(P)$ between $t = 0$ and $t = h$. Consider now the system (C.1) and multiply it by $2\mathbf{u}$ to find

$$\frac{\partial(\mathbf{u},\mathbf{u})}{\partial t} + \sum_{j=1}^{3} \frac{\partial(\mathbf{A}_j\mathbf{u},\mathbf{u})}{\partial x_j} = 0. \tag{C.5}$$

Integrating the last relation over the lens $L(h)$ we have

$$\int_{L(h)} \left(\frac{\partial(\mathbf{u},\mathbf{u})}{\partial t} + \sum_{j=1}^{3} \frac{\partial(\mathbf{A}_j\mathbf{u},\mathbf{u})}{\partial x_j} \right) \mathrm{d}x\,\mathrm{d}t = 0. \tag{C.6}$$

Applying the Gauss formula to the integral we obtain

$$\int_{S(h)} (\mathbf{u},\mathbf{u})\,\mathrm{d}x - \int_{S(0)} (\mathbf{u},\mathbf{u})\,\mathrm{d}x + \int_{\mathfrak{R}^*(h)} \frac{1}{|\nabla_x \Phi(x,t)|} \left(\left[\mathbf{I}\frac{\partial \Phi(x,t)}{\partial t} + \sum_{j=1}^{3} \mathbf{A}_j \frac{\partial \Phi(x,t)}{\partial x_j} \right] \mathbf{u}, \mathbf{u} \right) \mathrm{d}S = 0, \tag{C.7}$$

where $\Phi(x,t) \in \mathscr{C}^1(\mathbb{R}^3 \times [0,h])$ is the function such that the surface $\mathfrak{R}^*(h)$ and $(0 < h < T)$ is given by $\Phi(x,t) = 0$.

Using the proposition and the last relation we find

$$\|\mathbf{u}\|^2_{S(h)} \leqslant \|\mathbf{u}\|^2_{S(0)}. \tag{C.8}$$

This proves the lemma. □

The following uniqueness theorem follows from this lemma.

Theorem 14. *Under the same conditions as were imposed on the Lemma* 13, *if a solution* $\mathbf{u}(x,t)$ *of* (C.1) *vanishes on* $S(0)$ *it also vanishes on each surface* $S(h)$ *which forms with* $S(0)$ *a space-like lens.*

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